COMP 2804 - Winter 2025 Study Session Questions & Answers

April 23, 2025

1 Question 1

1.1 Question:

A flip in a bitstring is a pair of adjacent bits that are not equal. For example, the bitstring `010011` has three flips: The first two bits form a flip, the second and third bits form a flip, and the fourth and fifth bits form a flip.

- Determine the number of bitstrings of length 7 that have exactly 3 flips at the following positions: The second and third bits form a flip, the third and fourth bits form a flip, and the fifth and sixth bits form a flip.
- Let $n \geq 2$ and k be integers with $0 \leq k \leq n-1$. Determine the number of bitstrings of length n that have exactly k flips.

1.2 Explanation:

A **flip** in a bitstring is defined as a pair of adjacent bits that are not equal.

Part 1: Fixed Flip Positions in Bitstrings of Length 7

We are given that the flips must occur between:

- Positions 2 and 3,
- Positions 3 and 4,
- Positions 5 and 6.

Let the bitstring be denoted by:

$$b_1 b_2 b_3 b_4 b_5 b_6 b_7$$

A flip at position i means $b_i \neq b_{i+1}$. We are told that:

$$b_2 \neq b_3, \quad b_3 \neq b_4, \quad b_5 \neq b_6$$

From $b_2 \neq b_3$, $b_3 \neq b_4$ and $b_5 \neq b_6$, it follows the following by transitivity:

1. $b_1 == b_2$ since there is no flip between the two characters
2. $b_4 == b_5$ since there is no flip between the two characters
3. $b_6 == b_7$ since there is no flip between the two characters

Since each character depends on the next character, depending on the starting bit chosen, it will affect every other bit in the bitstring. This means that there are only two possible solutions (since there are only two possible bits for b_1 to be: (either 0 or 1).

Part 2: Bitstrings of Length n with Exactly k Flips

Let $n \geq 2$ and $0 \leq k \leq n-1$. A bitstring of length n has $n-1$ adjacent positions where a flip could occur.

To construct a bitstring with exactly k flips:

- Choose k out of the $n-1$ positions to place flips: $\binom{n-1}{k}$ choices,
- Choose the starting bit (0 or 1): 2 choices.

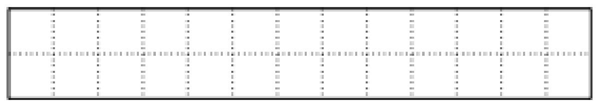
Therefore, the number of such bitstrings is:

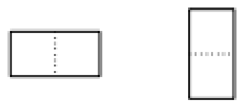
$$\boxed{2 \cdot \binom{n-1}{k}}$$

2 Question 2

Question:

4.24 Let $n \geq 1$ be an integer and consider a $2 \times n$ board B_n consisting of $2n$ square cells. The top part of the figure below shows B_{13}.

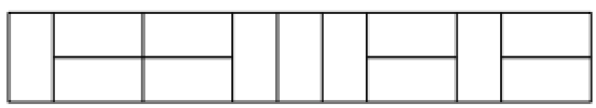




A *brick* is a horizontal or vertical board consisting of 2 square cells; see the bottom part of the figure above. A *tiling* of the board B_n is a placement of bricks on the board such that

- the bricks exactly cover B_n and
- no two bricks overlap.

The figure below shows a tiling of B_{13}.



For $n \geq 1$, let T_n be the number of different tilings of the board B_n. Determine the value of T_n, i.e., express T_n in terms of numbers that we have seen in this chapter.

2.1 Explanation:

Let $n \geq 1$ be an integer and consider a $2 \times n$ board B_n consisting of $2n$ square cells. A **brick** is either:

- a horizontal 1×2 brick (placed side-by-side), or
- a vertical 2×1 brick (stacked on top of each other).

Let T_n denote the number of different tilings of B_n using these bricks such that:

- the bricks exactly cover B_n, and
- no two bricks overlap.

Recurrence Relation

To determine T_n, we consider the structure of a tiling and analyze the last brick placed:

1. If the last brick is **vertical**, it covers the final column. The remaining board is B_{n-1}, which can be tiled in T_{n-1} ways.

2. If the last two bricks are **horizontal**, they must occupy the last two columns. The remaining board is B_{n-2}, which can be tiled in T_{n-2} ways.

Thus, we arrive at the recurrence:

$$T_n = T_{n-1} + T_{n-2}$$

This is exactly the recurrence relation for the **Fibonacci sequence**, as defined in **Section 4.2**.

3 Question 3

3.1 Question:

Algorithm DIGDIRT(n):

if $n = 1$:

Drink water for **one minute**;

return;

else

Shovel for n minutes;

DIGDIRT($\frac{n}{3}$);

DIGDIRT($\frac{n}{3}$);

You should assume that $n = 3^m$ for an integer m (or that n is always a multiple of 3).

Determine the number of minutes you drink water as a function of m.

3.2 Explanation:

Let $F(n)$ be the number of minutes that you drink water in the above algorithm on an input of n.
From running the algorithm, we know that $F(1) = 1$.
A call to $F(n)$ calls itself recursively twice, which means that we can calculate $F(n)$ by using the unfolding technique:

$$\begin{aligned} F(n) &= 2 \cdot F(n/3) \\ &= 2 \cdot 2 \cdot F(n/3^2) \\ &= 2^2 \cdot F(n/3^2) \\ &= 2^3 \cdot F(n/3^3) \\ &\vdots \\ &= 2^k \cdot F(n/3^k). \end{aligned}$$

The recursion ends at $F(n/3^k) = F(1)$, or when $n/3^k = 1$ or $n = 3^k$.

If we take $\log_3$ of both sides we have $k = \log_3 n$.

$$F(n) = 2^{\log_3 n} \cdot F(1) = 2^{\log_3 n}.$$

Now we need the solution in terms of m. Recall that $n = 3^m$, thus

$$2^{\log_3 n} = 2^{\log_3 3^m} = 2^m.$$

So we end up catching our breath for $\boxed{2^m}$ minutes.

4 Question 4

4.1 Question:

A standard deck consists of a set of 52 cards. There are 13 distinct ranks,

$$\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K\}.$$

There are 4 cards of each rank, one of each of the following 4 suits: $\{\heartsuit, \diamondsuit, \spadesuit, \clubsuit\}$. In traditional poker, each player is dealt 5 cards initially. Assume the dealer is using a single deck of 52 cards and deals you a hand of 5 cards uniformly at random.

Define the events:

- F = You are dealt 5 cards of the same suit (called a *flush*).
- D = You are dealt at least one diamond.

For this question, F includes all possible hands of 5 cards with the same suit.

1. What is $\Pr(F)$?
2. What is $\Pr(D)$?
3. What is the probability of getting a flush given that there is at least one diamond in your hand?

4.2 Explanation:

4.2.1 Answer for 1)

To compute the probability of getting a flush, we can use the standard probability formula:

$$\Pr(F) = \frac{|F|}{|S|} = \frac{\text{number of 5-card flushes}}{\text{total number of 5-card hands}}.$$

There are 4 suits as specified above, and $\binom{13}{5}$ flushes per suit, so:

$$\Pr(F) = \frac{4 \cdot \binom{13}{5}}{\binom{52}{5}} = \frac{4 \cdot 1287}{2598960} = \frac{5148}{2598960} = \frac{33}{16660}.$$

4.2.2 Answer for 2)

Let S be the sample space of "all the ways to be dealt five cards."
We can use the complement rule to find the solution easier than brute forcing it.
The complement of D is: the number of hands with *no diamonds*, which is $\left|\overline{D}\right| = \binom{39}{5}$:

$$\begin{aligned}\Pr(D) &= 1 - \Pr(\overline{D}) \\ &= 1 - \frac{|\overline{D}|}{|S|} \\ &= 1 - \frac{\binom{39}{5}}{\binom{52}{5}} \\ &= \frac{7411}{9520}.\end{aligned}$$

4.2.3 Answer for 3)

We can represent the probability that "we get a flush given that there is at least one diamond in your hand?" as $Pr(F|D)$.
From this definition, we can define the event $F \cap D$ as the event where we receive a flush consisting of all diamonds.
Using these two definitions, we can calculate $Pr(F|D)$ using the conditional probability formula from the textbook:

$$\begin{aligned}
\Pr(F \mid D) &= \frac{\Pr(F \cap D)}{\Pr(D)} \\
&= \frac{\frac{|F \cap D|}{|S|}}{\Pr(D)} \\
&= \frac{\frac{\binom{13}{5}}{\binom{52}{5}}}{\frac{7411}{9520}} \\
&= \frac{33}{\binom{52}{5}} \cdot \frac{9520}{7411} \\
&= \frac{33}{51877}.
\end{aligned}$$

5 Question 5

5.1 Question

In the *5-107 Lottery* you choose a set of 6 distinct integers $\{x_1, \dots, x_5, y\}$ from the set $\{1, 2, 3, \dots, 107\}$. $x_1, \dots, x_5$ are called your *main numbers* and y is your *bonus number*. On Friday night, the lottery machine draws a uniformly random 5-number subset $\{z_1, \dots, z_5\}$ from $\{1, \dots, 107\}$. You buy one lottery ticket with your favourite 6 numbers.

1. You win a **Big Jackpot** if $\{x_1, \dots, x_5\} = \{z_1, \dots, z_5\}$. What is the probability that you win the Big Jackpot?

2. You win a **Little Jackpot** if $|\{x_1, \dots, x_5\} \cap \{z_1, \dots, z_5\}| = 4$. What is the probability that you win a Little Jackpot?

5.2 Explanation:

5.2.1 Answer for 1)

The sample space for this question is the set S of all $\binom{107}{5}$ 5-element subsets of $\{1, \dots, 107\}$. An element in S represents the numbers $\{z_1, \dots, z_5\}$ chosen by the machine. This is a uniform sample space, so $\Pr(\omega) = 1/|S| = 1/\binom{107}{5}$ for each $\omega \in S$.

Let A be the event "you win the Big Jackpot." Then

$$\Pr(A) = \Pr(\{x_1, \dots, x_5\}) = \frac{1}{\binom{107}{5}}.$$

5.2.2 Answer for 2)

Let B be the event "you win a Little Jackpot" and, for each $i \in \{1, \dots, 5\}$, let B_i be the event

$$\{x_1, \dots, x_5\} \cap \{z_1, \dots, z_5\} = \{x_1, \dots, x_5\} \setminus \{x_i\}.$$

In words, B_i is the event "all your numbers matched except for x_i". The events $B_1, \dots, B_5$ are pairwise disjoint since in each event, the position of the matching number changes. Therefore, by Sum Rule:

$$\Pr(B) = \Pr\left(\bigcup_{i=1}^{5} B_i\right) = \sum_{i=1}^{5} \Pr(B_i),$$

Now we just need to figure out $\Pr(B_i) = |B_i|/|S| = |B_i|/\binom{107}{5}$. Without loss of generality, we focus on $\Pr(B_5)$.

In words, the event B_5 happens when $\{z_1, \dots, z_5\}$ is a 5-element subset of $\{1, \dots, 107\} \setminus \{x_5\}$ that contains $\{x_1, \dots, x_4\}$. In other words, $\{z_1, \dots, z_5\} = \{x_1, \dots, x_4, z_5\}$, where z_5 is any integer in the 102-element set $\{1, \dots, 107\} \setminus \{x_1, \dots, x_5\}$.

Therefore,

$$|B_5| = \binom{102}{1} = 102.$$

There is nothing special about x_5 in this argument, so we can generalize it for all B_i:

$$\Pr(B_1) = \Pr(B_2) = \dots = \Pr(B_5) = \frac{|B_5|}{\binom{107}{5}} = \frac{102}{\binom{107}{5}}.$$

Now we can calculate $Pr(B)$:

$$\Pr(B) = \Pr\left(\bigcup_{i=1}^{5} B_i\right) = \sum_{i=1}^{5} \Pr(B_i) = 5 \cdot \Pr(B_5) = \frac{510}{\binom{107}{5}}.$$

6 Question 6

6.1 Question:

In Blackjack you want a hand that totals as close to 21 as possible without going over. The dealer will start by dealing you two cards, then deal themselves one card, all face up. For this question, we will assume that the dealer is using a standard deck of 52 cards. All numbered cards have a value equal to their number. All face cards (King = K, Queen = Q, Jack = J) are worth 10. Aces = A are worth 1 or 11.

If you are dealt an Ace and a King as your first two cards (in either order), you have Blackjack and will win if the dealer does not also get Blackjack.

Let D be the event that you've been dealt a Blackjack. That is, D is the event that you are dealt one Ace and one of $10, J, Q$, or K of any suit and in either order.

Let E be the event that the dealer deals themselves an Ace. That is, E is the event that the dealer's first card is in the set $\{A\heartsuit, A\diamondsuit, A\clubsuit, A\spadesuit\}$.

1. What is $\Pr(D \cap E)$?
2. What is $\Pr(D \cup E)$?
3. Are the events D and E independent?

6.2 Explanation:

6.2.1 Answer to 1)

The sample set S is all the possible three-card combinations that could be dealt. There are 52 ways to choose the first card, 51 ways to choose the second card, and 50 ways to choose the third card.

$D \cap E$ is the event that the first three cards are A, 10, A. There are 4 ways to choose the first Ace, 3 ways to choose the second Ace, and 16 ways to choose a card worth 10.

$$\Pr(D \cap E) = \frac{|D \cap E|}{|S|} = \frac{16 \cdot 4 \cdot 3 \cdot 2}{52 \cdot 51 \cdot 50} = \frac{16}{5525} \approx 0.002896$$

6.2.2 Answer to 2)

Using the Inclusion/Exclusion formula, we can derive the following:

$$|D \cup E| = |D| + |E| - |D \cap E|$$

We compute:

- $|C| = 4 \cdot 16 \cdot 2 \cdot 50$ since there are 4 Aces, 16 cards worth 10, 2 permutations, and 50 remaining cards for the dealer.
- $|D| = 4 \cdot 51 \cdot 50$ since there are 4 Aces and 51 ways to choose first card, 50 for second.
- $|C \cap D| = 16 \cdot 4 \cdot 3 \cdot 2$ (from part 1 above)

Then:

$$\Pr(D \cup E) = \frac{|D| + |E| - |D \cap E|}{|S|} = \frac{4 \cdot 16 \cdot 2 \cdot 50 + 4 \cdot 51 \cdot 50 - 16 \cdot 4 \cdot 3 \cdot 2}{52 \cdot 51 \cdot 50} = \frac{2027}{16575} \approx 0.12$$

6.2.3 Answer to 3)

Using the independence formula definition from the textbook, is $\Pr(D \cap E) = \Pr(D) \cdot \Pr(E)$?

The answer is no, they are not independent since:

$$\Pr(D \cap E) = \frac{16}{5525}, \quad \text{and} \quad \Pr(D) \cdot \Pr(E) = \frac{(4 \cdot 16 \cdot 2 \cdot 50)(4 \cdot 51 \cdot 50)}{(52 \cdot 51 \cdot 50)^2} = \frac{32}{8619}$$

So in conclusion,

$$\Pr(D \cap E) \neq \Pr(D) \cdot \Pr(E)$$

Therefore, D and E are **not independent**.

7 Question 7

7.1 Question:

You go fishing in a lake. There are two types of fish (trout and pike) in the lake. Each time you cast your line, you catch a trout with probability 4/5 and you catch a pike with probability 1/5. You throw every fish you catch back in the water, so the result of each cast is independent of which types of fish you caught on previous casts. You stop fishing once you've caught at least one trout and at least one pike.

What is the expected number of casts you make before you stop fishing?

7.2 Explanation:

The sample space is:

$$S = \{T^n P : n \geq 0\} \cup \{P^n T : n \geq 0\}.$$

Where the sequence of letters indicates the sequence of fish caught, T for trout and P for pike. The probability function for this question is:

$$\Pr(T^n P) = (4/5)^n(1/5), \quad \Pr(P^n T) = (1/5)^n(4/5).$$

Let the number of casts be the random variable X defined by:

$$X(T^n P) = X(P^n T) = n + 1 \quad \text{for each } n \geq 0.$$

The reason the random variable X is $n+1$ and not n is we guarantee have n casts before the terminating cast and then we have the 1 terminating cast, which adds to $n+1$ total casts.

Then the expected value of X is:

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{\omega \in S} \Pr(\omega) \cdot X(\omega) \\
&= \sum_{n=0}^{\infty} \Pr(T^n P) \cdot X(T^n P) + \sum_{n=0}^{\infty} \Pr(P^n T) \cdot X(P^n T) \\
&= \sum_{n=0}^{\infty} (4/5)^n (1/5)(n+1) + \sum_{n=0}^{\infty} (1/5)^n (4/5)(n+1) \\
&= \sum_{n=1}^{\infty} (4/5)^{n-1} (1/5) n + \sum_{n=1}^{\infty} (1/5)^{n-1} (4/5) n \\
&= (1/5) \sum_{n=0}^{\infty} (4/5)^n n + (4/5) \sum_{n=0}^{\infty} (1/5)^n n \\
&= (1/5) \cdot \frac{1}{(1-4/5)^2} + (4/5) \cdot \frac{1}{(1-1/5)^2} \text{ (using geometric sequence substitution)} \\
&= (1/5) \cdot \frac{1}{(1/5)^2} + (4/5) \cdot \frac{1}{(4/5)^2} \\
&= \frac{1}{1/5} + \frac{1}{4/5} \\
&= 5 + \frac{5}{4} = \frac{21}{4}.
\end{aligned}$$

8 Question 8

8.1 Question:

Each month you are sent one of n different brands of pizza uniformly at random, and independently of previous orders.

Define the random variable X to be the total number of distinct brands of pizza that you receive. Determine the expected value $\mathbb{E}(X)$ of X.

8.2 Explanation:

We can define the indicator random variable:

$$X_i = \begin{cases} 1 & \text{if pizza brand } i \text{ is chosen at least once} \\ 0 & \text{otherwise} \end{cases}$$

Since X_i is an indicator random variable, $\mathbb{E}(X_i) = \Pr(X_i = 1)$, which is the probability that, out of 12 orders, pizza brand i is selected at least once. We use the complement rule:

$$\Pr(X_i = 1) = 1 - \Pr(X_i = 0),$$

where $X_i = 0$ is the scenario where pizza brand i is never selected in any of the orders.

Let a_j be the event that pizza brand i is selected for order j. Then:

$$\Pr(a_j) = \frac{1}{n}, \quad \text{so} \quad \Pr(\overline{a}_j) = \frac{n-1}{n}.$$

Since each order is independent and we assume that there are 12 orders in 1 year, the probability that pizza brand i is never chosen over 12 months is:

$$\Pr(X_i = 0) = \Pr(\overline{a}_1 \wedge \overline{a}_2 \wedge \cdots \wedge \overline{a}_{12}) = \left(\frac{n-1}{n}\right)^{12}.$$

So,

$$\Pr(X_i = 1) = 1 - \left(\frac{n-1}{n}\right)^{12}.$$

By Linearity of Expectation:

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}(X_1 + X_2 + \cdots + X_n) \\ &= \mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}(X_n) \\ &= \sum_{j=1}^{n} \Pr(X_j = 1) \\ &= \sum_{j=1}^{n} \left(1 - \left(\frac{n-1}{n}\right)^{12}\right), \text{ subbing in value} \\ &= n\left(1 - \left(\frac{n-1}{n}\right)^{12}\right), \text{ from summation definition} \end{aligned}$$

9 Question 9

9.1 Question:

Let D be a standard deck of cards.

For a card $c \in D$, let the value of $X(c)$ be equal to the value of the card. So 1 if the rank is A, 2 if the card is 2, ..., and 10 if the rank is in $\{J, Q, K\}$.

Assume you are dealt a standard hand of 5 cards.

Let the folllowing be random variables:

$$V = \text{The sum of the values of the cards in your hand,}$$
$$F = \text{The number of face cards (that is, } J, Q, \text{ or } K\text{) in your hand.}$$

1. What are $\mathbb{E}(V)$ and $\mathbb{E}(F)$?
2. Are V and F independent random variables?

9.2 Explanation:

9.2.1 Answer to 1)

Let $C_1, \ldots, C_5$ be the values of the individual cards in your hand. Then by definition:

$$\mathbb{E}(V) = \sum_{i=1}^{5} \mathbb{E}(C_i).$$

All $\mathbb{E}(C_i)$ are equal by symmetry, so:

$$\mathbb{E}(C_i) = \sum_{i=1}^{9} i \cdot \frac{1}{13} + 10 \cdot \frac{4}{13} = \frac{85}{13}, \quad \Rightarrow \quad \mathbb{E}(V) = 5 \cdot \frac{85}{13} = \frac{425}{13}.$$

Now define:

$$F_i = \begin{cases} 1 & \text{if card } i \text{ is a face card} \\ 0 & \text{otherwise} \end{cases}$$

Then:

$$\mathbb{E}(F) = \sum_{i=1}^{5} \mathbb{E}(F_i) = 5 \cdot \frac{3}{13} = \frac{15}{13}.$$

9.2.2 Answer to 2)

For calculating whether to not V and F are independent events, we just have to use the indepdent random variable formula. To prove that these two variables are not independent, we just have to find a scenario where for some value of these events, the following formula is not correct: $Pr(X = x \cap Y = y) = Pr(X = x) \cdot Pr(Y = y)$

For example, assume the event $V = 6$ occurs when the hand consists of 4 aces and a 2. Also assume the event $F = 1$ occurs when the hand has exactly 1 face card.

Both of these events have a non-zero probability individually (i.e it is possible for this event to occur), but their intersection has probability 0 (you cannot have a hand with a value of 6 if you have a face card in your hand since a face card has a value of 10 by default). In other words, $Pr(V = 6 \cap F = 1) \neq Pr(V = 6) \cdot Pr(F = 1)$.

Thus, the independent random variables V and F are **not independent**.