COMP 2804 - Winter 2019 Midterm Solutions

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1 Question 1

1. Consider bitstrings of length 13. The positions in these strings are numbered as $1, 2, 3, \ldots, 13$. How many such bitstrings have the property that all bits at the odd positions are equal?

    (a) 32
    (b) 64
    (c) 128
    (d) 256

Since we have 13 positions in the string and every odd position is equal, we can "block" those spots essentially (since no different elements can go in these spots). We can visualize the bitstring like the following:

- blocked - blocked - blocked - blocked - blocked - blocked -

So with the open spots left, we know that in a bitstring that any part of the pitstring can be a 1 or a 2, which means the open spots can either contain a 1 or a 2. Since they are 6 open spots, we can calculate the number of combinations of valid bitstrings using the product rule:

1. Valid bitstrings $= 2 * 2 * 2 * 2 * 2 * 2 * 2 = 2^7 = 128$.

The only answer that matches 128 is c)

2 Question 2

2. Consider permutations of the set $\{a, b, c, d, e, f, g\}$ that do not contain bge (in this order).
   How many such permutations are there?

    (a) $7! - 6!$
    (b) $7! - 5!$
    (c) $7! - 4!$
    (d) $7! - 3!$

We see from the question answers that this is going to be a complement rule question since $7!$ corresponds to all the permutations that can be generated by the set. Since we are applying the complement rule, we need to find out how many permutations contain (bge) in that order.

To produce permutations using (bge), we can use a visual to help us see what positions are taken up by bge and what positions are left to put elements. Let's say hypothetically, that we decide to put the (bge) group at the beginning 3 spots of the permutation:

"b" "g" "e" - - - - -

From this, we can see that there are 5 open positions, and we have 5 remaining letters left. Knowing this, we can find out the invalid permutations:

1. # invalid permutations $= 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

From this answer, the amount of valid permutations is $7! - 5!$ using the complement rule, which corresponds to answer b).

■

3 Question 3

3. Consider strings of length 15, where each character is a lowercase letter or an uppercase letter.
   How many such strings contain at least one lowercase letter and at least one uppercase letter?

   (a) $52^{15} - 26^{15}$
   (b) $52^{15} - 2 \cdot 26^{15}$
   (c) $52^{15} - 3 \cdot 26^{15}$
   (d) None of the above

Again, similar to the last question, we need to use a complement rule to find the answer (given that all the answers are in the form of a complement rule).

In this case, the opposite of "at least one lowercase letter and at least one uppercase letter" becomes "no lowercase letters or no uppercase letters". Let's tackle each of these cases individually:

For the case of no lowercase letters, we know that there are 26 letters in the alphabet so if we have no lowercase letters, this means we can only have uppercase letters. There are 15 spots in the string, so using product rule: 26^{15} combinations.

For the case of no uppercase letters, its the same but now we can only have lowercase letters. So similarly, we end up with 26^{15} combinations.

So, if we apply the complement rule, we end up with the following: $52^{15} - (26^{15} + 26^{15})$ which simplifies to $52^{15} - 2 \cdot 26^{15}$, which corresponds to answer b)

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4 Question 4

4. Let $n \geq 8$ be an even integer and let k be an integer with $2 \leq k \leq n/2$. Consider k-element subsets of the set $S = \{1, 2, \ldots, n\}$. How many such subsets contain at least two even numbers?

(a) $\binom{n}{k} - \binom{n/2}{k-1} - \frac{n}{2} \cdot \binom{n/2}{k}$

(b) $\binom{n}{k} - \binom{n/2}{k-1} - \frac{n}{2} \cdot \binom{n/2}{k-1}$

(c) $\binom{n}{k} - \binom{n/2}{k} - \frac{n}{2} \cdot \binom{n/2}{k}$

(d) $\binom{n}{k} - \binom{n/2}{k} - \frac{n}{2} \cdot \binom{n/2}{k-1}$

Given the formatting of the answers in this question, we know that the answer is going to involve Complement Rule since we see that there is some sort of "master set" which is being subtracted from.

In the question, we see that we want to find "subsets containing at least two even numbers", which if we apply complement rule/de morgan's law to that claim, we end up with: "subsets containing strictly less than two even numbers".

The specific "groups" that fit this claim are that we either have k-element subsets with 0 even numbers or we have k-element subsets with 1 even numbers. Let's tackle each of these individually:

# of 0 even number subsets: Given that we have n numbers in the set S to choose from, if we have 0 even numbers, we essentially remove half of the pool of numbers to choose from. Since we have to choose k elements to form a k-element subset, the # of 0 even number subsets are $\binom{n/2}{k}$.

# of 1 even number subsets: If we have to choose 1 even number from n numbers, this means that we have $\binom{n/2}{1}$ amount of ways to choose that 1 even number, which simplifies down to $\frac{n}{2}$. For the remaining spots in the k-element subset, they have to be odd numbers (since we have already chosen our 1 even number), which means that there are $\binom{n/2}{k-1}$ choices for the odd numbers. Applying Product Rule, the # of 1 even number subsets is $\frac{n}{2} \cdot \binom{n/2}{k-1}$.

Now that we know the invalid subsets (the ones that we do not want), we can apply complement rule to find out the subsets we do want to find (the ones with at least two even numbers). The universal set in this case is $\binom{n}{k}$ since that would be selecting a k-element subset with no restrictions. Applying Complement Rule would give the following:

1. Compelemnt Rule $= |U| \backslash A$, definition of complement rule

2. Complement Rule $= \binom{n}{k} - (\frac{n}{2} + \frac{n}{2} \cdot \binom{n/2}{k-1})$, subbed in the values we calculated above

3. Complemnt Rule $= \binom{n}{k} - \frac{n}{2} - \frac{n}{2} \cdot \binom{n/2}{k-1}$, simplified

The only answer that matches our answer is 4d)

5 Question 5

5. In a group of 100 students,

- 37 students like beer,

- 18 students like cider,

- 55 students do not like beer and do not like cider.

How many students like beer and cider?

(a) 8

(b) 9

(c) 10

(d) 11

Let A represent the set of students who like beer = 37
Let B represent the set of students who like cider = 18

Let $\overline{A} \cap \overline{B}$ represent the set of students who do not like beer and do not like cider = 55
Let $A \cap B$ represent the set of students who like beer and like cider (this is what we want to find) = ?

Given that we know the value of $\overline{A} \cap \overline{B}$, we can calculate the opposite of $\overline{A} \cap \overline{B}$, using complement rule, which will be helpful later on:

1. $|U| = |\overline{A} \cap \overline{B}| + |\overline{\overline{A} \cap \overline{B}}|$, from definition of universal set

2. $|U| = |\overline{A} \cap \overline{B}| + |A \cup B|$, simplified by using DeMorgan's Law

3. $100 = 55 + |A \cup B|$, subbed in values we know

4. $45 = |A \cup B|$, solved for $|A \cup B|$

Now that we know $|A \cup B|$, we can rearrange the formula for Inclusion-Exclusion principle to find $A \cap B$:

1. $|A \cup B| = |A| + |B| - |A \cap B|$, definition of Inclusion-Exclusion principle for 2 sets

2. $45 = 37 + 18 - |A \cap B|$, subbed in values we know

3. $|A \cap B| = 37 + 18 - 45$, rearranged to solve for $|A \cap B|$

4. $|A \cap B| = 10$, simplified

With an answer of 10 for $A \cap B$, the matching answer is 5c)

6 Question 6

6. Let $n \geq 1$ be an integer. A group of n students write an exam. Each student receives a grade, which is an element of the set $\{A, B, C, D, F\}$.
What is the minimum value for n, such that there must be at least four students who receive the same grade?

   (a) 14

   (b) 15

   (c) 16

   (d) 17

This is an example of a Pigeonhole Principle question, but the solution I'm giving is not really as rigorous as a normal Pigeonhole Principle solution is. From the question, we know that we want to find the minimum number of students such that there must be at least four students who receive the same grade.

So, if we just start creating a list of students and give them a grade, and basically try and avoid giving repeated grades for as long as possible, we get the following:

1. Student # 1: A

2. Student # 2: B

3. Student # 3: C

4. Student # 4: D

5. Student # 5: F

6. Student # 6: A

7. Student # 7: B

8. Student # 8: C

9. Student # 9: D

10. Student # 10: F

11. Student # 11: A

12. Student # 12: B

13. Student # 13: C

14. Student # 14: D

15. Student # 15: F

At this point, the next student that we add will have to select a grade that 3 other students have. This means that no matter what, when we add the 16th student, there will be at least four students who receive the same grade. This means that when $n = 16$,

This means that the minimum value for n, such that there must be at least four students who receive the same grade is 16, which corresponds to answer 6c)

□

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7 Question 7

7. Consider 17-element subsets of the set $\{1, 2, 3, \ldots, 45\}$.
   How many such subsets have the property that the largest element in the subset is equal to 30?

   (a) $\binom{29}{16}$

   (b) $\binom{29}{17}$

   (c) $\binom{30}{16}$

   (d) $\binom{30}{17}$

So we know that the largest element of the subset we're creating has to be equal to 30. This means that we cannot choose a number larger than 30 for the remaining 16 spots.

For the remaining 16 spots, we can only choose numbers between 1-29 (since number 30 has already been chosen, and we cannot choose a number larger than 30) and there are 16 more numbers to complete the 17-element subset. This means that are $\binom{1}{1} \cdot \binom{29}{16}$ subsets which have the property we want, which simplifies to $\binom{29}{16}$.

This corresponds to answer 7a)

$\square$

8 Question 8

8. Let $n \geq 4$ be an integer. What does

$$3 \cdot \binom{n}{3} + 6n \cdot \binom{n}{2} + n^3$$

count?

(a) The number of ways to choose an ordered triple

$$(\text{beer bottle}, \text{cider bottle}, \text{wine bottle})$$

in a set consisting of n beer bottles, n cider bottles, and n wine bottles.

(b) The number of ways to choose a 3-element subset of a set consisting of n beer bottles, n cider bottles, and n wine bottles.

(c) The number of ways to choose 3 elements (with repetitions allowed) in a set consisting of n beer bottles, n cider bottles, and n wine bottles.

(d) None of the above.

This question tests your theoretical comprehension of binomial coefficients, rather than straight computation. While on an exam, it is possible to sub in values of n and check which definitions match up when subbing in real numbers for n, it is best (IMO) to solve these questions using pure theory rather than brute-forcing (since if the question is designed properly, it will take a long time to brute-force to eliminate multiple-choice options)

Looking at the definition, the only one that matches is 8b) and here is the explanation why:

When we select a 3-element subset and we have 3 types of elements to put into the subset (n beer bottles, n cider bottles and n wine bottles), we have 3 different options for the subsets:

1. 3 of the same element

2. 2 of the same element and one different element

3. 1 of each element

If we sum up all of those options, we will have found the number of ways to choose a 3-element subset of a set consisting of n beer bottles, n cider bottles and n wine bottles. Here are the calculations for each of those options:

3 of the same element: If we choose 3 of the same element and each type has n options, then that means that the number of ways is $\binom{n}{3}$ for each option. There are 3 options so in, total there are $\binom{n}{3} + \binom{n}{3} + \binom{n}{3} = 3 \cdot \binom{n}{3}$ number of ways.

2 of the same element and one different element: If we choose 2 of one element, there are $\binom{n}{2}$ ways to do that. For the one different element, there will always be 2 options for that (since there are 3 types in total), which means that there are $2 \cdot \binom{n}{1} = 2n$. We have 3 different types that we can choose to select the "2 of the same element and one different element" which means that the total # of ways to create "2 of the same element and one different element" subsets is: $3 \cdot (2n \cdot \binom{n}{2}) = 6n \cdot \binom{n}{2}$.

1 of each element: For this combination, if we select one element from each of the types and each of the types has n values, then in total the subset will contain (beer bottle, cider bottle, wine bottle). To create this subset, we can choose 1 value from each element that corresponds to $\binom{n}{1} \cdot \binom{n}{1} \cdot \binom{n}{1} = n \cdot n \cdot n = n^3$

combinations.

Now that we have calculated each of the individual options of subsets, we can use the Sum Rule, to aggregate all the subsets together, which results in the following expression: $3 \cdot \binom{n}{3} + 6n \cdot \binom{n}{2} + n^3$, which matches the expression given in the question, which means that answer 8b) is correct.

□

9   Question 9

9. Let $n \geq 4$ be an even integer and let k be an integer with $1 \leq k \leq n/2$. Consider strings consisting of n characters, such that

- each character is an element of $\{a, b, c\}$,

- the number of a's is equal to k, and

- each a is at an even position.

How many such strings are there?

(a) $\binom{n/2}{k} \cdot 2^{n-k}$

(b) $\binom{n/2}{k} \cdot 2^{n/2}$

(c) $\binom{n}{k} \cdot 2^{n-k}$

(d) $\binom{n}{k} \cdot 2^{n/2}$

To calculate the number of valid strings, we need to look at the properties of the strings in the question. We know that we have to select k number of $a's$ and that each a is at an even position.

Since n is specified to be an even integer, then the amount of even positions in the string is $n/2$. This means to select the a elements, there are $\binom{n/2}{k}$ options.

For the remaining positions, they can only be b or c, which means that there are 2 options, and there are $n - k$ spots remaining since k spots have been taken up by the $a's$. This means that there are 2^{n-k} combinations for the remaining strings.

We need to combine both of these combinations to form a string of n length using the product rule, which results in $\binom{n/2}{k} \cdot 2^{n-k}$, which corresponds to answer 9a)

10   Question 10

10. A bitstring is called 00-*free*, if it does not contain two 0's next to each other. In class, we have seen that for any $m \geq 1$, the number of 00-free bitstrings of length m is equal to the $(m+2)$-th Fibonacci number f_{m+2}.

What is the number of 00-free bitstrings of length 77 that have 0 at position 59. (The positions are numbered $1, 2, \ldots, 77$.)

(a) $f_{17} \cdot f_{57}$

(b) $f_{18} \cdot f_{58}$

(c) $f_{19} \cdot f_{59}$

(d) $f_{20} \cdot f_{60}$

This question is best usually answered with a drawing, which I will provide and then explain the logic behind it:
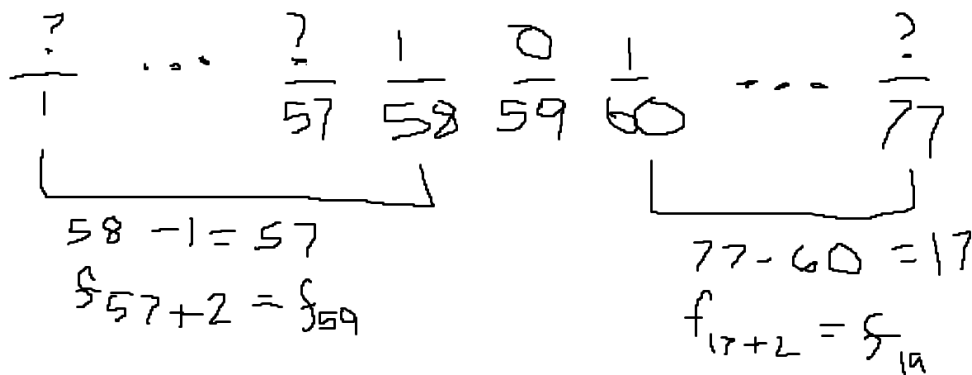
There is a bit to unpack here but here's the logic behind the image, but I'll unpack the steps I took to find the two Fibonacci bitstrings involved:

1. First, I created a bitstring of length 77, with the position numbers under each spot in the bitstring. I didn't write out all of the entries and only drew the key entries (the 1st spot, spots 57-60, and the 77th/last spot)

2. I put the number 0 at position 59, as specified in the question description. From this, I deduced that to keep the bitstring 00-free, the numbers around the 0 at position 59 had to be 1, since if we put a 0 beside the number at position 59, it would create a 00, which would mean that the string is not 00-free.

3. From this, I calculated the length of the bitstrings to the right of position 60 and left of position 58, since those bitstrings are the ones that we can have any numbers as long as the bitstring remains 00-free. The string on the right has a length of 17 and the string on the left has a string of 57.

4. From the definition of a 00-free bitstring in the question, we know that the Fibonacci corresponding to the 00-free bitstring is bitstring length $+ 2$. So to convert the bitstrings into their corresponding fibonacci numbers, I just added 2 to their length, which results in a Fibonacci number of 59 for the left bitstring and a Fibonacci number of 19 for the right bitstring.

5. Since we want to find the number of 00-free bitstrings of length 77, we need to use Product Rule to combine the left and right bitstrings, which results in: $f_{19} \cdot f_{59}$. This answer corresponds to answer 10c).

□

11 Question 11

11. The function $f : \mathbb{N} \to \mathbb{N}$ is recursively defined as follows:

$$\begin{aligned} f(0) &= 6, \\ f(n) &= 4 \cdot f(n-1) + 2^n \quad \text{if } n \geq 1. \end{aligned}$$

Which of the following is true for all integers $n \geq 0$?

(a) $f(n) = 6 \cdot 4^n - 2^n$

(b) $f(n) = 7 \cdot 4^n - 2^n$

(c) $f(n) = 8 \cdot 4^n - 2^{n+1}$

(d) None of the above

To solve this problem, I am going to use an inductive proof to prove that answer b) is the correct answer:

1. Base Case: If we plug in $n = 0$ into answer b), we get:

   (a) $f(0) = 7 \cdot 4 - 2^0$, subbed in 0 into closed form for b)

   (b) $f(0) = 7 \cdot 1 - 1$, simplified

   (c) $f(0) = 6$, simplified

   We now know that the closed form for b) passes for its base case

2. Inductive Hypothesis: Assume that the function f, the closed form is: $f(n-1) = 7 \cdot 4^{n-1} - 2^{n-1}$

3. Inductive Step: We will now prove that the closed form works for n:

   (a) $f(n) = 4 \cdot f(n-1) + 2^n$, recursive recurrence definition of f

   (b) $f(n) = 4 \cdot [7 \cdot 4^{n-1} - 2^{n-1}] + 2^n$, invoked inductive hypothesis

   (c) $f(n) = (4 \cdot 7 \cdot 4^{n-1}) - (4 \cdot 2^{n-1}) + 2^n$, expanded RHS

   (d) $f(n) = 7 \cdot 4^n - (4 \cdot 2^{n-1}) + 2^n$, invoked exponent rules to add the 4 into 4^{n-1}

   (e) $f(n) = 7 \cdot 4^n - (2^2 \cdot 2^{n-1}) + 2^n$, rewriting 4 as 2^2

   (f) $f(n) = 7 \cdot 4^n - 2^{n+1} + 2^n$, invoked exponent rules to add 2^2 into 2^{n-1}

   (g) $f(n) = 7 \cdot 4^n - 2^n$, added $-2^{n+1} + 2^n$ together We have now ended up with the closed form in answer b) using the recurrence, which means we have proved that the closed form is valid for all integers $n \geq 0$.

Since we have proved with induction that answer b) is the closed form of the recurrence, answer 11b) is the correct answer for this question.

12 Question 12

12. Consider strings of characters, where each character is an element of the set $\{a, b, c\}$. Such a string is called *awesome*, if it does not contain aa, and does not contain aba, and does not contain abb.

   For any integer $n \geq 1$, let A_n be the number of awesome strings of length n.

   Which of the following is true for any integer $n \geq 4$?

   (a) $A_n = A_{n-1} + 2 \cdot A_{n-2} + 2 \cdot A_{n-3}$.

   (b) $A_n = A_{n-1} + 2 \cdot A_{n-2} + A_{n-3}$.

   (c) $A_n = 2 \cdot A_{n-1} + A_{n-2} + 2 \cdot A_{n-3}$.

   (d) $A_n = 2 \cdot A_{n-1} + A_{n-2} + A_{n-3}$.

This is an example of a recursive bitstring problem (which is common among midterms), so here is generally how I approach solving these types of questions.

In this question, we want to find the number of ways of creating an awesome string. So here are all the ways we can create an awesome string:

1. We start with the letter c, and then concatenate an awesome string of size A_{n-1} to it to from an awesome string of size n

2. We start with the letter b, and then concatenate an awesome string of size A_{n-1} to it to from an awesome string of size n

3. We start with the letters ac, and then concatenate an awesome string of size A_{n-2} to it to from an awesome string of size n

4. We start with the letters abc, and then concatenate an awesome string of size A_{n-3} to it to from an awesome string of size n

We can't start the string with any other combination of letters, since there is no guarantee that in those cases, we can produce an awesome string.

We can use the sum rule to aggregate all these cases together, to find all the ways to produce an awesome string of length n:

1. $A_n = $ all the ways of forming an awesome string of length n, by definition of an awesome string

2. $A_n = A_{n-1} + A_{n-1} + A_{n-2} + A_{n-3}$, adding together all the different recursive answers from above for valid awesome string starting combinations

3. $A_n = 2 \cdot A_{n-1} + A_{n-2} + A_{n-3}$, simplified

The answer in the question that matches this is the answer 12d)

13 Question 13

13. We consider binary $2 \times n$ matrices, i.e., matrices with 2 rows and n columns, in which each entry is 0 or 1. A column in such a matrix is called a $\frac{1}{1}$-*column*, if both bits in the column are 1.

For any integer $n \geq 1$ and integer k with $0 \leq k \leq n$, let $M(n, k)$ be the number of binary $2 \times n$ matrices

- that do not contain any $\frac{1}{1}$-column and

- contain exactly k many 1's.

Which of the following is true for all integers $n \geq 2$ and k with $1 \leq k \leq n - 1$?

(a) $M(n, k) = M(n - 1, k) + M(n - 1, k - 1)$

(b) $M(n, k) = M(n - 1, k) + 2 \cdot M(n - 1, k - 1)$

(c) $M(n, k) = M(n, k - 1) + M(n - 1, k - 1)$

(d) $M(n, k) = M(n, k - 1) + 2 \cdot M(n - 1, k - 1)$

This question is a bit confusing, so I'm going to throw a diagram to help explain what a $\frac{1}{1}$ column is:

Column 1	Column 2
1	0
1	1

So, I drew a 2x2 matrix, and labeled the first column as Column 1 and the second column of the matrix as Column 2. In this case, Column 1 is a $\frac{1}{1}$ column since all values in the column are a 1. Column 2 is not a $\frac{1}{1}$ column since it has a 0 as a value in ihe column.

To find all binary 2 x n matrices that are $\frac{1}{1}$ column free and contain exactly k many 1's, we can denote a recursive algorithm for this.

So, imagine that we are at column n in the matrix, and we want to allocate 1's and 0's for the 2 entries in this column. We have two options for the values we put in this column:

1. We put two zeroes in the column

2. We put one zero and one "1" in the column

Let's try and find the recursive definition for each of these options:

We put two zeroes in the column: If we put two zeroes in the column, we don't use any of our k 1's that we're allowed to put, but we do fill in 1 column, which means the n value is going to decrease. This results in the following recursive call: $M(n - 1, k)$

We put one zero and one "1" in the column: To put a zero and a 1 in the column, there are two ways to do it:

1
0

or

0
1

In either case, we use one of our k-allocated 1's and we also fill in a column. Since we have two ways of doing this, this results in the following recursive call: $2 \cdot M(n-1, k-1)$.

Now that we know our two ways of recursively creating a valid matrix (according to question definitions), we can combine both of them to find all the ways of creating a valid matrix of size n with k-allocated 1's: $M(n,k) = M(n-1,k) + 2 \cdot M(n-1,k-1)$.

The only answer that matches this is answer 13b)

$\blacksquare$

14 Question 14

14. Consider the recursive algorithm HELLOWORLD, which takes as input an integer $n \geq 0$:

> **Algorithm** HELLOWORLD(n):
> **if** $n = 0$ or $n = 1$
> **then** print *Hello World*
> **else if** n is a multiple of 3
>   **then** HELLOWORLD($n/3$);
>   print *Hello World*;
>   HELLOWORLD($2n/3$)
>   **else** HELLOWORLD($n + 1$)
>   **endif**;
> **endif**

Which of the following is correct?

(a) For any integer $n \geq 0$, algorithm HELLOWORLD(n) terminates.

(b) There exists an integer $n \geq 0$, for which algorithm HELLOWORLD(n) does not terminate.

(c) All of the above.

(d) None of the above.

The indentation is a little bit scuffed in this question, but the answer to this is b), and here is why (using the power of deduction). We know from the question that answers c) and d) cannot be right:

1. Answer c) cannot be right, because both answers a) and b) cannot be true at the same time. You cannot both say that the algorithm terminates and say that the algorithm does not terminate for all numbers ≥ 0.

2. Answer d) also cannot be right, because an algorithm has to either terminate or cannot terminate (unless you have the Schordinger's cat of algorithms, which does not apply here).

So our options are either a) or b), but we can prove that there is an integer for which HELLOWORLD(n) doesn't terminate. Let's try running the algorithm with $n = 3$:

```
HELLOWORLD(3) -> HELLOWORLD(1) -> terminates
              -> HELLOWORLD(2) -> HELLOWORLD(3) -> HELLOWORLD(1) -> terminates
                                               -> HELLOWORLD(2) -> ....
```

The explanation for the text above is that when we have an n that is a multiple of 3, we have two recursive calls, one of which terminates and the other which ends up resulting in the number 2. The recursive call with the number 2 is problematic, because the "else HELLOWORLD(n+1)" line, causes the algorithm to recursively increase n until we get a value that is a multiple of 3, and we repeat forever.

So for this question, the answer is 14b)

15 Question 15

15. The Carleton Computer Science Society (CCSS) is organizing their annual Saint Patrick's Day party. The CCSS has bought three types of drinks:

- Porterhouse Brewing Co. Oyster Stout.

- Guinness Extra Stout.

- Magners Original Irish Cider.

There is an unlimited supply for each of these types.
There are 75 students at the party, and each of them gets one drink, which is chosen uniformly at random from these three types.
Let A be the event

$$A = \text{"exactly 50 students get Magners Original Irish Cider".}$$

What is $\Pr(A)$?

(a) $\frac{\binom{75}{50} \cdot 2^{25}}{3^{75}}$

(b) $\frac{3^{75}}{\binom{75}{50} \cdot 2^{25}}$

(c) $\frac{\binom{75}{50}}{3^{75}}$

(d) $\frac{\binom{75}{50} \cdot 3^{25}}{3^{75}}$

To solve this question, we need to find out how many ways there are for "exactly 50 students to get Magners Original Cider". There are 75 students to choose from, and 50 of them need to get a certain type of drink guaranteed, which results in $\binom{75}{50}$.

For the remaining 25 people, they also need a drink and they can only choose from two drinks: Porterhouse Brewing Co. Oyster Stout and Guinness Extra Stout. This means that there are $\binom{25}{25} \cdot 2^{25}$ ways for this to happen, which simplifies to 2^{25}

To get sample space A, we need to combine our two answers above using the Product Rule, which results in: $\binom{75}{50} \cdot 2^{25}$

Now that we know the sample space of event A, we can find the probability of A. The universal sample space, $|S|$, corresponds to 3^{75}, since this assumes that each person has the option to select from the 3 drinks, and there are 75 students.

1. $Pr(A) = \frac{|A|}{|S|}$, definition of probability with event A

2. $Pr(A) = \frac{\binom{75}{50} \cdot 2^{25}}{3^{75}}$, subbed in the values we know

The answer above corresponds to answer 15a)

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16   Question 16

16. Consider a standard 6-sided fair die. We roll this die 8 times. Let A be the event

$$A = \text{``there are at least two 5's in the sequence of 8 rolls''}.$$

What is $\Pr(A)$?

(a) $1 - \frac{8 \cdot 5^7}{6^8}$

(b) $1 - \frac{6^8}{5^8 + 8 \cdot 5^7}$

(c) $1 - \frac{5^8 + 8 \cdot 5^7}{6^8}$

(d) $1 - \frac{8^5 + 8 \cdot 7^5}{8^6}$

Given the format of the answers in the question, we know that we will have to apply the Complement Rule to find the correct answer.

Applying the complement rule to the claim in the question, we turn "there are at least two 5's in the sequence of 8 rolls" into "there are less than two 5's in the sequence of 8 rolls". If we have less than two 5's in the sequence of 8 rolls, two options can be viable:

1. We roll zero 5's in the sequence of 8 rolls

2. We roll one 5 in the sequence of 8 rolls

Let's tackle each of the case individually:

We roll zero 5's in the sequence of 8 rolls: To get zero 5's in the sequence of 8 rolls with a 6-sided dice, this means we have to roll either a $\{1,2,3,4,6\}$ as our roll, 8 times in a roll. At each roll, there are 5 options and we roll 8 times, and the total sample space is 6^8, which results in the following probability: $\frac{5^8}{6^8}$.

We roll one 5 in the sequence of 8 rolls: To only roll one 5 in a sequence of 8 rolls, we first choose the specific roll where we get the number 5 on the dice. There are 8 possible rolls to choose from to get the number 5. For the 7 remaining rolls, they have to be a number other than 5, which means that for those rolls, there are 5 valid options. In total, it results in the following probability: $\frac{\binom{8}{1} \cdot 5^7}{6^8}$ which simplifies to $\frac{8 \cdot 5^7}{6^8}$.

Now that we know all the probabilities for Complement Rule, we can apply the Complement Rule to get our answer:

1. $\Pr(A) = 1$ - (the probabilities of the rolls we do not want)

2. $\Pr(A) = 1 - (\frac{5^8}{6^8} + \frac{8 \cdot 5^7}{6^8})$, added in the values we know

3. $\Pr(A) = 1 - \frac{5^8 + 8 \cdot 5^7}{6^8}$, simplified fractions

Our final answer corresponds to answer 16c)

17 Question 17

17. This midterm has 17 questions. For each question, four options are given, exactly one of which is correct.

    Assume that you answer each question, by choosing one of the four options uniformly at random.

    Let A be the event

    $$A = \text{``you answer at least 16 questions correctly''}.$$

    What is $\Pr(A)$?

    (a) $\frac{4^{17}}{52}$

    (b) $\frac{51}{4^{17}}$

    (c) $\frac{49}{4^{17}}$

    (d) $\frac{52}{4^{17}}$

To answer this question, we need to find the number of ways of answering at least 16 questions correctly. There is a total of 17 questions on the midterm in the question, so answering at least 16 questions correctly can be one of two outcomes:

1. getting 16 questions correct and 1 wrong

2. getting 17 questions correct and 0 wrong

Let's tackle each of these outcomes:

Getting 16 questions correct and 1 wrong: To get 16 questions correct and 1 wrong, we have to first select the 1 question to get wrong. There are 17 questions to choose from, and 3 answers are wrong in each question. For the 16 questions that we get correct, there is only 1 correct answer for each question, which means in total, there is a $\frac{\binom{17}{1} \cdot 3 \cdot \binom{16}{16} \cdot 1}{4^{17}}$ chance to get 16 questions correct and 1 question wrong. This simplifies down to $\frac{17 \cdot 3 \cdot 1 \cdot 1}{4^{17}} = \frac{51}{4^{17}}$.

Getting 17 questions correct and 0 wrong: To get all 17 questions correct, there is only 1 way to do this, which is to select the 1 correct option 17 times in a row, which results in a probability of $\frac{\binom{17}{17} \cdot 1}{4^{17}}$, which simplifies to $\frac{1}{4^{17}}$.

Now that we know the two separate probabilities, we can use the Sum Rule to add the two cases together to find the number of ways to answer at least 16 questions correctly: $\frac{51}{4^{17}} + \frac{1}{4^{17}} = \frac{52}{4^{17}}$, which corresponds to answer 17d)

$\blacksquare$